NONISOTHERMAL SEMIBOUNDED PLANE JET OF INCOMPRESSIBLE VISCOUS FLUID

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A self-similar solution for the temperature in a plane semibounded jet was obtained in [1]. The self-similarity temperature constant $\alpha_T = 1/2$ was found by using the integral conservation condition

$$\int_0^\infty uT\left(\int_0^y u\,dy\right)dy = \text{const},$$

which in general is not satisfied (this integral is independent of the longitudinal coordinate only on the assumption of similarity of the velocity and temperature profiles at Prandtl numbers close to unity). In this paper self-similar solutions for different Prandtl numbers are constructed, and it is shown that the self-similarity constant depends on the Prandtl number.

The problem of a plane laminar semibounded jet moving along a solid wall can be written, in the boundary-layer approximation, in the dimensionless form

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{\partial^2 u}{\partial y^2}, \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0; \tag{1}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \frac{1}{\Pr}\frac{\partial^2 T}{\partial y^2}; \tag{2}$$

$$u = v = 0 \text{ when } y = 0, \; u = 0 \text{ when } y \to \infty; \tag{3}$$

$$T = 0 \text{ when } y = 0, \; T = 0 \text{ when } y \to \infty, \tag{4}$$

where x, $yR^{-3/4}$ are rectangular Cartesian coordinates (x, y are the inner coordinates in the asymptotic expansion in terms of the small parameter $R^{-1}$ corresponding to the boundary-layer limit); $R = (L_m I_m/\rho_m)^{1/3}/\mu_m$ is an analog of the Reynolds number; $uR^{1/2}$, $vR^{-1/4}$ are the longitudinal and transverse velocity components; T is the difference between the temperatures at the given point and at infinity; $\Pr = c_{pm}\mu_m/\lambda_m$ is the Prandtl number. In converting to dimensionless form, it was assumed that the scales of the heat capacity at constant pressure $c_{pm}$, the dynamic viscosity $\mu_m$, the thermal conductivity $\lambda_m$, the length $L_m$, the density $\rho_m$, the temperature $T_m$, and the Akatnov invariant

$$I_m = \rho_m V_m^3 L_m^2 \int_0^\infty u^2\left(\int_0^y u\,dy\right)dy$$

are prescribed. As a velocity scale we chose

$$V_m = (I_m/\rho_m L_m^2)^{1/3}.$$

For problem (1)-(4) we should impose initial conditions at $x = x_0$, but within the scope of this paper we will consider only self-similar solutions. For closure of the dynamic problem (1), (3), we formulate the Akatnov conservation condition

$$\int_0^\infty u^2\left(\int_0^y u\,dy\right)dy = 1. \tag{5}$$

As regards the heat problem, it follows from its form that the obtained solutions of (2), (4) are accurate to a constant factor, the value of which will not be assigned at present.

Problem (1)-(5) admits the self-similar solution

$$u(x, y) = x^{-1/2}F'(\eta), \; v(x, y) = x^{-3/4}V(\eta), \; T(x, y) = x^{\alpha_T}\Theta(\eta), \; y = \eta x^{3/4}; \tag{6}$$

here and henceforth the dash denotes the derivative with respect to $\eta$; F, V, $\Theta$, and $\eta$ are self-similar variables. Substituting (6) in (1)-(5), we obtain for the dynamic problem

$$4F''' + F''F + 2F'^2 = 0, \; F = F' = 0 \text{ when } \eta = 0, \; F' = 0 \text{ when } \eta \to \infty; \tag{7}$$

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$$\int_0^\infty FF'^2 d\eta = 1, \tag{8}$$

and for the heat problem

$$\Theta'' + \Pr\left(\frac{1}{4}F\Theta' - \alpha_T F'\Theta\right) = 0,\ \Theta = 0 \text{ when } \eta = 0,\ \Theta = 0 \text{ when } \eta \to \infty, \tag{9}$$

and to find the transverse velocity we can use the equation

$$V = (1/4)(3\eta F' - F).$$

The linear homogeneous equation with homogeneous boundary conditions (9) represents a singular boundary-value eigenvalue problem; the role of the eigenvalue here is played by the parameter $\alpha_T$. Problem (9) has probably a denumerable set of solutions. From this set we selected the solution of positive sign from physical considerations. Numerical calculations showed that it was unique.

Problems (7), (9) are invariant to the transformation

$$F \to C_1 F,\quad \Theta \to C_2\Theta,\quad \eta \to C_1^{-1}\eta, \tag{10}$$

i.e., instead of the Akatnov variant (8) we can use for problem (7) the nontriviality condition in the form

$$F'' = 1 \text{ when } \eta = 0, \tag{11}$$

and for normalization of the solution for the temperature we select the condition

$$\Theta' = 1 \text{ when } \eta = 0. \tag{12}$$

Thus, using (10) we reduced the boundary-value problem (7), (8) to the Cauchy problem (7), (11). Solving the problems (7), (11), (9), (12), we can then, using the invariant properties of (10), normalize the solution according to (8) or in some other way. Problem (7), (11), (9), and (12) was solved numerically for different Prandtl numbers. Figure 1 shows a family of plots of $\Theta$ as a function of $\eta$; the curve numbers correspond to the Prandtl number. The profile of $F'(\eta)$, characterizing the longitudinal velocity (6), is the same as the profile of the dimensionless temperature at Pr = 1. The maximum of the temperature profile when Pr → 0 is shifted towards higher coordinates $\eta$ (i.e., moves away from the wall for a fixed coordinate x). The temperature decreases more slowly with increase in $\eta$ (or y for fixed x) at low than at high Prandtl numbers. When Pr ≫ 1 the temperature profile becomes much "narrower" than the dynamic profile (Pr = 1) and, conversely, when Pr → 0 the "thickness" of the temperature profile tends to infinity. The Akatnov invariant for problem (7) and (11) is

$$\int_0^\infty u^2\left(\int_0^y u\,dy\right)dy = 7.49.$$

The value of this constant will be required if the solution has to be normalized in accordance with (8).

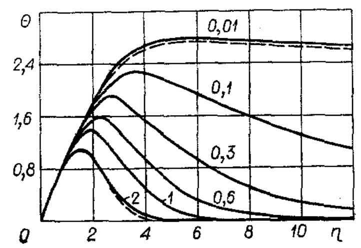


Fig. 1

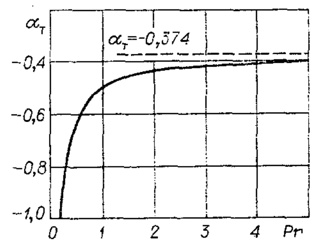


Fig. 2

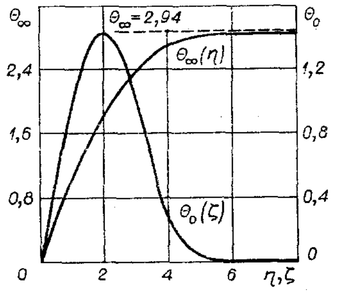


Fig. 3

Figure 2 shows the dependence of the self-similarity temperature constant $\alpha_T$ [the eigenvalue of problem (9)] on the Prandtl number. It is apparent that $\alpha_T \to$ const when $\Pr \to \infty$, and $\alpha_T \to -\infty$ when $\Pr \to 0$. It follows from Eqs. (6) that when $\Pr \to 0$ ($\alpha_T \to -\infty$) the difference in temperatures in the jet and at infinity decays rapidly (equivalent of "high" thermal conductivity), and when $\Pr \to \infty$ the maximum temperature in the jet decays independently of the Prandtl number, since in this case $\alpha_T \to$ const.

We will explain the limit behavior of solutions of the problem (7), (11), (9), and (12) in cases of arbitrarily large and arbitrarily small values of the Prandtl number. In the investigation we will use the ideas and terminology of perturbation methods [2].

Let

$$\varepsilon = 1/\Pr \to 0. \tag{13}$$

Then the asymptotic expansion, associated with the inner limiting process, when

$$\xi = \eta/\delta(\varepsilon) \text{ is fixed, } \delta(\epsilon) \to 0 \text{ when } \epsilon \to 0, \text{ can be constructed in the form} \tag{14}$$

$$\Theta(\varepsilon, \eta) = \nu_0(\varepsilon)\Theta_0(\xi) + \ldots, \quad \alpha_T(\varepsilon) = \alpha_0 + \ldots, \quad F(\eta) = F_0\delta^2\xi^2 + \ldots, \tag{15}$$

where only the zero (main) terms of the expansions are written; $\delta(\epsilon)$ is a quantity characterizing the scale of the inner variable $\xi$; $\nu_0(\epsilon)$ is the zero term of the asymptotic sequence for $\Theta(\varepsilon, \eta)$. It follows from the form of problem (9), (12), (7), (11) that

$$F_0 = 1/2, \quad \nu_0 = \delta = \varepsilon^{1/3}. \tag{16}$$

Substituting the series (15) in problem (9), (12), we obtain at the limit (13), (14)

$$\begin{gathered} \frac{d^2\Theta_0}{d\xi^2} = -\frac{1}{8}\xi^2\frac{d\Theta_0}{d\xi} + \alpha_0\xi\Theta_0, \ \Theta_0 = 0, \\ \Theta_0' = 1 \quad \text{for} \quad \xi = 0, \\ \Theta_0 = 0 \quad \text{for } \xi \to \infty. \end{gathered} \tag{17}$$

Figure 3 shows the numerically obtained solution of the eigenvalue problem (17). It follows from the calculations that the eigenvalue $\alpha_0 = -0.374$. The problem (17) arising in the zeroth approximation in $\epsilon$ satisfies all the boundary conditions (at zero and at infinity), i.e., in this case there is no need to construct the outer solution and to effect matching, and the expansion of the solutions of problem (7), (11), (9), (12) in form (15) is uniformly valid for $\eta$.

The dashed lines in Fig. 1 approximate solutions corresponding to the zeroth approximation of expansion (15). It is apparent that when Pr = 2 the agreement of the exact and approximate solutions is satisfactory, i.e., even at moderately high Prandtl numbers we can eliminate explicitly the similarity criterion Pr, and for the description of nonisothermal flow we can use problem (17), which does not include the Prandtl number ($\epsilon$).

We now consider the case $\Pr \ll 1$. Let

$$\varepsilon = \Pr \to 0. \tag{18}$$

For the solutions of problem (7), (11), (9), and (12) we first consider the inner limit, where

$$\xi = \eta/\Delta(\varepsilon) \quad \text{is fixed,} \quad \Delta \to \infty \text{ when } \varepsilon \to 0, \tag{19}$$

and $\Delta(\epsilon)$ is a quantity characterizing the scale of the outer variable $\xi$. The dynamic problem (7), (11) has an analytic solution [1], which tends exponentially to a constant when $\eta \to \infty$:

$$F = F_\infty + \mathrm{TST}(\eta), \quad F' = \mathrm{TST}(\eta) \text{ when } \eta \to \infty \quad (F_\infty = 4.16), \tag{20}$$

i.e., in Eq. (9) at the outer limit the term containing F' drops out. The remaining terms of Eq. (9) will be of the same order if at the outer limit (19) we put

$$\Delta(\varepsilon) = \varepsilon^{-1}. \tag{21}$$

Then the expansions associated with the limit (18), (19), (21) can be constructed in the form

$$\Theta(\eta, \varepsilon) = \tilde{\nu}_\infty(\varepsilon)\tilde{\Theta}_\infty(\xi) + \ldots, \quad F(\eta) = F_\infty + \ldots \tag{22}$$

Substituting expansions (22) and bearing (18) – (21) in mind, we obtain the problem in the zeroth approximation in $\epsilon$

$$\frac{d^2\tilde{\Theta}_\infty}{d\xi^2} = -\frac{1}{4}F_\infty\frac{d\tilde{\Theta}_\infty}{d\xi}, \quad \tilde{\Theta}_\infty = 0 \quad \text{when } \xi \to \infty. \tag{23}$$

The solution of problem (23) is

$$\tilde{\Theta}_\infty = \tilde{C}_\infty \exp\left(-\frac{1}{4} F_\infty \xi\right). \quad (24)$$

The quantities $\tilde{\nu}_\infty(\epsilon)$ in (22) and $\tilde{C}_\infty$ in (24) can be determined subsequently from the condition for matching the outer and inner expansions.

To construct an expansion that is valid close to the limit $\eta = 0$, we formulate an inner limiting process when (25) $\eta$ is fixed when $\epsilon \to 0$, and the expansion of the solutions of the problem (9), (12) at the limit (18), (25) has the form

$$\Theta(\varepsilon, \eta) = \Theta_\infty(\eta) + \ldots, \alpha_T(\varepsilon) = \alpha_\infty \varepsilon^{-1} + \ldots \quad (26)$$

Then, substituting (26) in problem (9), (12) and having the definitions (18), (25) in mind, we obtain the problem in the zeroth approximation in $\epsilon$

$$\Theta_\infty'' = \alpha_\infty F' \Theta_\infty, \quad \Theta_\infty = 0, \quad \Theta_\infty' = 1 \quad \text{when } \eta = 0. \quad (27)$$

We note that in the formulation of the inner limiting process (18), (25) and the construction of the asymptotic expansion (26) we were guided by the fact that: 1) the zeroth-approximation problem contains the eigenvalue $\alpha_\infty$; 2) the limit (18), (25) is "characteristic" (the boundary-layer limit [2]).

When $\eta \to \infty$ problem (27) [see Eq. (20)] has the obvious asymptotic solution

$$\Theta_\infty = C_\infty + C_3\eta + \mathrm{TST}(\eta). \quad (28)$$

It follows from the form of the solution at the outer limit (24) that in Eq. (28) we must put $C_3 = 0$. Thus, for the problem at the inner limit (27) we can impose the condition

$$\Theta_\infty' = 0 \quad \text{when } \eta \to \infty. \quad (29)$$

Problem (27), (29) is an eigenvalue problem. Figure 3 shows the numerically calculated function $\Theta_\infty$ of $\eta$. When $\eta \to \infty$, $\Theta_\infty = 2.94$. The eigenvalue from calculation is $\alpha_\infty = -0.125$. Matching the inner and outer expansions at a limit intermediate between the inner and outer we can obtain for the outer expansion

$$\tilde{\nu}_\infty(\varepsilon) = 1, \ \tilde{C}_\infty = C_\infty = 2.94.$$

Adding the outer and inner expansions and subtracting the common part, we obtain an approximation uniformly valid in $\eta$

$$\Theta(\varepsilon, \eta) = \Theta_\infty(\eta) + C_\infty\left[\exp\left(-\frac{1}{4} F_\infty \varepsilon\eta\right) - 1\right]. \quad (30)$$

The dashed lines in Fig. 1 show the approximate solution when Pr = 0.01. The satisfactory agreement with the exact solution is obvious. Thus, at low Prandtl numbers too the similarity criterion Pr can be eliminated and Eq. (30) will be valid for description of the temperature profile.

For numerical integration we used the Runge–Kutta method. The eigenvalues were selected from the condition that the temperature at infinity [or its derivative for problem (27), (29)] is zero by the segment bisection method.

## LITERATURE CITED


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